Curl of a Vector field:

Let $\vec{F}(x,y,z)$ be a vector point function,

when $\vec{F}$ is operated vectorially by the operator ∇, we get a vector quantity known as curl of the vector, is defined as $\nabla \times \vec{F}$.

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

<u>Note</u>: 1) If $\vec{F}$ is a constant function, then curl $\vec{F} = 0$.

2) If c is a scalar then curl$(c\vec{F}) = c$ curl $\vec{F}$.

3) Curl of a vector field is a vector quantity.

4) The symbol $\nabla \times \vec{F}$ does not represent the cross product of ∇ and $\vec{F}$ in the sense of cross product of two vectors, because ∇ is just a differential operator and not a vector.

5) Physical meaning: If $\vec{v}$ is the velocity of a particle in a rigid body rotating about a fixed axis with a uniform angular velocity, $\frac{1}{2}$ Curl $\vec{v}$ is equal to the angular velocity of the body.

Also curl $\vec{F}$ is associated physically with rotation or spin of fluid. Hence it is also written as rot $\vec{F}$.

Irrotational vector:

A vector $\vec{F}$ is said to be irrotational if Curl $\vec{F} = 0$. A constant vector field is a trivial example of an irrotational vector.

Properties:

1) If $\vec{v}$ is a constant vector, show that
$\nabla \times \vec{v} = \vec{0}$.

Proof: Given $\vec{v}$ is a constant vector.

Consider, $\nabla \times \vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \vec{v}$

$$= \frac{\partial \vec{v}}{\partial x} \times \hat{i} + \frac{\partial \vec{v}}{\partial y} \times \hat{j} + \frac{\partial}{\partial z}\vec{v} \times \hat{k}$$

$$= 0 \times \hat{i} + 0 \times \hat{j} + 0 \times \hat{k}$$

$$= \vec{0}$$

2) If $\vec{F}$ and $\vec{G}$ are two vector functions, then

$$\nabla \times (\vec{F} + \vec{G}) = \nabla \times \vec{F} + \nabla \times \vec{G}$$

Proof: ~~Let~~ $\cancel{\vec{F} = F_1 \hat{i}}$

$$\nabla \times (\vec{F} + \vec{G}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \times (\vec{F} + \vec{G})$$

$$= \hat{i} \times \frac{\partial}{\partial x}(\vec{F} + \vec{G}) + \hat{j} \times \frac{\partial}{\partial y}(\vec{F} + \vec{G}) + \hat{k} \times \frac{\partial}{\partial z}(\vec{F} + \vec{G})$$

$$= \hat{i} \times \left(\frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{G}}{\partial x} \right) + \hat{j} \times \left(\frac{\partial \vec{F}}{\partial y} + \frac{\partial \vec{G}}{\partial y} \right) + \hat{k} \times \left(\frac{\partial \vec{F}}{\partial z} + \frac{\partial \vec{G}}{\partial z} \right)$$

$$= \left[\hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \left(\frac{\partial \vec{F}}{\partial z} \right) \right] + \left[\hat{i} \times \frac{\partial \vec{G}}{\partial x} + \hat{j} \times \frac{\partial \vec{G}}{\partial y} + \hat{k} \times \frac{\partial \vec{G}}{\partial z} \right]$$

$$= (\nabla \times \vec{F}) + (\nabla \times \vec{G})$$

3) For any scalar field ϕ, show that

Curl $\nabla\phi = 0$ i.e; $\nabla \times (\nabla\phi) = 0$

Proof: Given ϕ is a scalar function,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla \times \nabla\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= i\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right] - \hat{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \hat{k}$$

$$\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right]$$

$$= 0$$

Conservative Vectors:

A vector field $\vec{F}$ is said to be conservative vector if there exists a scalar function ϕ such that $\vec{F} = \nabla\phi$, then ϕ is called a scalar potential of $\vec{F}$.

W.k.T $\nabla \times \nabla\phi = 0$ from this it follows that if $\vec{F}$ is a conservative vector then curl $\vec{F} = 0$, thus every conservative vector is an irrotational vector.

Problems:

1) If $\vec{a}$ is a constant vector, find curl$(\vec{r} \times \vec{a})$

Solution: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{r} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= (a_3y - a_2z)\hat{i} - (xa_3 - a_1z)\hat{j} + (a_2x - a_1y)\hat{k}$$

$$= (a_3y - a_2z)\hat{i} + (a_1z - a_3x)\hat{j} + (a_2x - a_1y)\hat{k}$$

$$\text{curl}(\vec{r}\times\vec{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3y-a_2z & a_1z-a_3x & a_2x-a_1y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(a_2x-a_1y) - \frac{\partial}{\partial z}(a_1z-a_3x)\right] - \hat{j}\left[\frac{\partial}{\partial x}(a_2x-a_1y) - \frac{\partial}{\partial z}(a_3y-a_2z)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(a_1z-a_3x) - \frac{\partial}{\partial y}(a_3y-a_2z)\right]$$

$$= \hat{i}\left[-a_1 - a_1\right] - \hat{j}\left[a_2+a_2\right] + \hat{k}\left[-a_3 - a_3\right]$$

$$= -2a_1\hat{i} - 2a_2\hat{j} - 2a_3\hat{k}$$

$$= -2(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$= -2\vec{a}$$

2) If $\vec{F} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$ find curl $\vec{F}$

Solution: $\vec{F} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(2xz)\right] + \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2y)\right]$$

$$= \hat{i}\left[2z + 2x\right] - \hat{j}\left[0-0\right] + \hat{k}\left[-2z - x^2\right]$$

$$= (2z+2x)\,\hat{i} - (2z+x^2)\,\hat{k}$$

3) Verify $\text{div Curl } \vec{F} = 0$ if $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$

<u>Solution:</u> $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$

To find Curl $\vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \hat{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= (x-x)\,\hat{i} - (y-y)\,\hat{j} + (z-z)\,\hat{k}$$

$$= 0 //$$

$div(curl\ \vec{F}) = div(0)$

$= 0$//

4) Prove that $div(curl\ \vec{F}) = 0$ i.e; $\nabla \cdot (\nabla \times \vec{F}) = 0$

<u>solution:</u> Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

To find $curl\ \vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{j} + \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]\hat{k}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$div(curl\ \vec{F}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot$$

$$\left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\right]$$

$$= \frac{\cancel{\partial^2 F_3}}{\cancel{\partial x \partial y}} - \frac{\cancel{\partial^2 F_2}}{\cancel{\partial x \partial z}} + \frac{\cancel{\partial^2 F_1}}{\cancel{\partial y \partial z}} - \frac{\cancel{\partial^2 F_3}}{\cancel{\partial y \partial x}} + \frac{\cancel{\partial^2 F_2}}{\cancel{\partial z \partial x}} - \frac{\cancel{\partial^2 F_1}}{\cancel{\partial z \partial y}}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

$$= \underline{0}$$

5) If $\vec{F} = x^2 y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$

find $\text{Curl}(\text{Curl}\,\vec{F})$ or $\nabla \times (\nabla \times \vec{F})$.

<u>Solution:</u>